

## Non-Perturbative Zero Modes in the Kraichnan Model for Turbulent Advection

Omri Gat, Victor S. L'vov, Evgenii Podivilov and Itamar Procaccia

*Department of Chemical Physics, The Weizmann Institute of Science, Rehovot 76100, Israel*

The anomalous scaling behavior of the  $n$ -th order correlation functions  $\mathcal{F}_n$  of the Kraichnan model of turbulent passive scalar advection is believed to be dominated by the homogeneous solutions (zero-modes) of the Kraichnan equation  $\hat{\mathcal{B}}_n \mathcal{F}_n = 0$ . Previous analysis found zero-modes in perturbation theory in a small parameter. We present non-perturbative analysis of the simplest (non-trivial) case of  $n = 3$  and compare the results with the perturbative predictions.

The Kraichnan model of turbulent passive scalar advection [1] has attracted enormous attention recently, [2–6] being the first non-trivial model of turbulent statistics in which the phenomenon of multi-scaling seems understandable by analytic methods. The model is for a scalar field  $T(\mathbf{r}, t)$  which satisfies the equation of motion

$$\frac{\partial T(\mathbf{r}, t)}{\partial t} + \mathbf{u}(\mathbf{r}, t) \cdot \nabla T(\mathbf{r}, t) = \kappa \nabla^2 T(\mathbf{r}, t) + \xi(\mathbf{r}, t). \quad (1)$$

Here  $\xi(\mathbf{r}, t)$  is a Gaussian white random force,  $\kappa$  is the diffusivity and the driving field  $\mathbf{u}(\mathbf{r}, t)$  is chosen to have Gaussian statistics, and to be “fastly varying” in the sense that its time correlation function is proportional to  $\delta(t)$ . The statistical quantities that one is interested in are the many point correlation functions

$$\mathcal{F}_{2n}(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_{2n}) \equiv \langle\langle T(\mathbf{r}_1, t) T(\mathbf{r}_2, t) \dots T(\mathbf{r}_{2n}, t) \rangle\rangle, \quad (2)$$

where double pointed brackets denote an ensemble average with respect to a stationary statistics of the forcing *and* the statistics of the velocity field. One of Kraichnan’s major results [2] is an exact differential equation for this correlation function,

$$\left[ -\kappa \sum_{\alpha} \nabla_{\alpha}^2 + \hat{\mathcal{B}}_{2n} \right] \mathcal{F}_{2n}(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_{2n}) = \text{RHS}. \quad (3)$$

The operator  $\hat{\mathcal{B}}_{2n} \equiv \sum_{\alpha > \beta}^{2n} \hat{\mathcal{B}}_{\alpha\beta}$ , and  $\hat{\mathcal{B}}_{\alpha\beta}$  are defined by

$$\hat{\mathcal{B}}_{\alpha\beta} \equiv \hat{\mathcal{B}}(\mathbf{r}_{\alpha}, \mathbf{r}_{\beta}) = h_{ij}(\mathbf{r}_{\alpha} - \mathbf{r}_{\beta}) \partial^2 / \partial r_{\alpha,i} \partial r_{\beta,j}, \quad (4)$$

where the “eddy-diffusivity” tensor  $h_{ij}(\mathbf{R})$  is given by

$$h_{ij}(\mathbf{R}) = h(R) [(\zeta_h + d - 1) \delta_{ij} - \zeta_h R_i R_j / R^2],$$

and  $h(R) = H(R/\mathcal{L})^{\zeta_h}$ ,  $0 \leq \zeta_h \leq 2$ . Here  $\mathcal{L}$  is some characteristic outer scale of the driving velocity field. The scaling properties of the scalar depend sensitively on the

scaling exponent  $\zeta_h$  that characterizes the  $R$  dependence of  $h_{ij}(\mathbf{R})$  and that can take values in the interval  $[0, 2]$ . Finally, the RHS in Eq.(3) is known explicitly, but is not needed here. The reason is that it was argued that the solutions of this equation for  $n > 1$  are dominated by the homogeneous solutions (“zero-modes”), in the sense that deep in the inertial interval the inhomogeneous solutions are negligible compared to the homogeneous one. Also, it was claimed that in the inertial interval one can neglect the Laplacian operators in Eq.(3), and remain with the simpler homogeneous equation  $\hat{\mathcal{B}}_{2n} \mathcal{F}_{2n} = 0$ .

Having exact differential equations for  $\mathcal{F}_{2n}$  allowed Kraichnan to announce a mechanism for anomalous scaling [2]. Assuming that the physical solutions are scale invariant one needs to examine the scaling (or homogeneity) exponent  $\zeta_{2n}$  of  $\mathcal{F}_{2n}$  which is defined by  $\mathcal{F}_{2n}(\lambda \mathbf{r}_1, \lambda \mathbf{r}_2 \dots \lambda \mathbf{r}_{2n}) = \lambda^{\zeta_{2n}} \mathcal{F}_{2n}(\mathbf{r}_1, \mathbf{r}_2 \dots \mathbf{r}_{2n})$  if such a solution exists. One expects it to exist in the inertial range, i.e. all the separations  $r_{ij}$  satisfy  $\eta \ll r_{ij} \ll L$  where  $\eta$  and  $L$  are the inner and outer scales respectively. It is known [1] that for  $\mathcal{F}_2$  such a solution exists with  $\zeta_2 = 2 - \zeta_h$ . If one solves for these exponent for  $n > 1$ , one can understand, at least in this simple model, what are the mechanisms for deviations from the predictions of dimensional analysis, with possible insight also for the Navier-Stokes problem. In searching methods for computing these important exponents, there emerged two basic strategies. One strategy considered the differential equation in the “fully unfused” regime in which all the separations between the coordinates are in the inertial range. In this case even in the simplest case of  $n = 2$  the function  $\mathcal{F}_4$  depends on six independent variables (for dimensions  $d > 2$ ), and one faces a formidable analytic difficulty for exact solutions. Accordingly, several groups considered perturbative solutions in some small parameter, like  $\zeta_h$  [3] or the inverse dimensionality  $1/d$  [4]. The rationale for this approach is that at  $\zeta_h = 0$  and  $d \rightarrow \infty$  one expects “simple scaling” with  $\zeta_{2n} = n\zeta_2$ . The exponent  $\zeta_4$ , and later also the set  $\zeta_{2n}$ , were computed as a function of  $\zeta_h$  near these simple scaling limits. Another strategy considered the differential equation in the “fully fused” regime, in which the correlation function degenerates to the structure function  $S_{2n}(R) = \langle\langle [T(\mathbf{r} + \mathbf{R}) - T(\mathbf{r})]^{2n} \rangle\rangle$ . In this approach there is an enormous simplification in having only one variable, but one loses information in the process of fusion. The lost information was supplemented [2] by a yet underived conjecture about the properties of conditional

averages, leading at the end to a close-form calculation of the exponents  $\zeta_{2n}$  for arbitrary dimension and values of  $\zeta_h$ . The results of the two strategies are not in agreement. Even though numerical simulations [5] and also experiments [7,8] lend support to the assumption used in the second strategy and to its computed values of  $\zeta_{2n}$ , there remains an important mystery as to why the two approaches reach such different conclusions. The aim of this Letter is to explore non-perturbative calculations of the zero modes and their exponents, to shed further light on this issue.

Our strategy is to solve exactly, eigenfunctions included, the homogeneous equation satisfied by the 3<sup>rd</sup> order correlation function  $\mathcal{F}_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ . Note that in Kraichnan's model all the odd-order correlation functions  $\mathcal{F}_{2n+1}$  are zero because of symmetry under the transformation  $T \rightarrow -T$ . This symmetry disappears for example [9] if the random force  $\xi(\mathbf{r}, t)$  is not Gaussian (but  $\delta$ -correlated in time), and in particular if it has a non-zero third order correlation

$$\mathcal{D}_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \equiv \int dt_1 dt_2 \langle \xi(\mathbf{r}_1, t_1) \xi(\mathbf{r}_2, t_2) \xi(\mathbf{r}_3, 0) \rangle. \quad (5)$$

With such a forcing the third order correlator is non-zero, and it satisfies the equation

$$\hat{\mathcal{B}}_3 \mathcal{F}_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \mathcal{D}_3, \quad \hat{\mathcal{B}}_3 \equiv \hat{\mathcal{B}}_{12} + \hat{\mathcal{B}}_{13} + \hat{\mathcal{B}}_{23}. \quad (6)$$

This equation pertains to the inertial interval and accordingly we neglected the Laplacian operators. We also denoted  $\mathcal{D}_3 = \lim_{\alpha\beta \rightarrow 0} \mathcal{D}_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ . The solution of this equation is a sum of inhomogeneous and homogeneous contributions, and below we study the latter. We will focus on scale invariant homogeneous solutions which satisfy  $\mathcal{F}_3(\lambda \mathbf{r}_1, \lambda \mathbf{r}_2, \lambda \mathbf{r}_3) = \lambda^{\zeta_3} \mathcal{F}_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ . We refer to these as the “zero modes in the scale invariant sector”. We note that the scaling exponent of the *inhomogeneous* scale invariant contribution can be read directly from power counting in Eq.(6) (leading to  $\zeta_3 = \zeta_2$ ). Any different scaling exponent can arise only from homogeneous solutions that do not need to balance the constant RHS. In addition, note that scale-invariant zero-modes arise not only due to the omission of the diffusive terms from Eq. (6), but also as a result of the omission of the boundary conditions for large separation (at the outer scale  $L$ ). The smooth connection to either small or large scales must ruin scale invariance. The scale invariant solutions of Eq.(6) live in a projective space whose dimension is lowered by unity compared to the most general form; These solutions do not depend on three separations but rather on two dimensionless variables that are identified below. It will be demonstrated how boundary conditions arise in this space for which the operator  $\hat{\mathcal{B}}_3$  is neither positive nor self-adjoint.

Equation (6) is also invariant under the action of the  $d$  dimensional rotation group  $\text{SO}(d)$ , and under permutations of the three coordinates. Here we seek solutions

in the scalar representation of  $\text{SO}(d)$ , where the solution depends on the 3 separations  $r_{12}$ ,  $r_{23}$  and  $r_{31}$  only. We transform coordinates to the variables  $x_1 = |\mathbf{r}_2 - \mathbf{r}_3|^2$ ,  $x_2 = |\mathbf{r}_3 - \mathbf{r}_1|^2$ ,  $x_3 = |\mathbf{r}_1 - \mathbf{r}_2|^2$ . The triangle inequalities in the original space are equivalent to the condition

$$2(x_1 x_2 + x_2 x_3 + x_3 x_1) \geq x_1^2 + x_2^2 + x_3^2. \quad (7)$$

The advantage of the new coordinates is that the inequality (7) describes a circular cone in the  $x_1, x_2, x_3$  space whose axis is the line  $x_1 = x_2 = x_3$  and whose circular cross section is tangent to the planes  $x_1 = 0$ ,  $x_2 = 0$  and  $x_3 = 0$ . This cone can be parameterized by three new coordinates  $s, \rho, \phi$ :

$$x_n = s\{1 - \rho \cos[\phi + (2\pi/3)n]\}, \\ 0 \leq s < \infty, \quad 0 \leq \rho \leq 1, \quad 0 \leq \phi \leq 2\pi. \quad (8)$$

The  $s$  coordinate measures the overall scale of the triangle defined by the original  $\mathbf{r}_i$  coordinates, and configurations of constant  $\rho$  and  $\phi$  correspond to similar triangles. The  $\rho$  coordinate describes the deviation of the triangle from the equilateral configuration ( $\rho = 0$ ) up to the physical limit of three collinear points attained when  $\rho = 1$ ;  $\phi$  does not have a simple geometric meaning.

The transformation of the linear operator  $\hat{\mathcal{B}}_3$  to the new coordinates is straightforward, and produces a second order linear partial differential operator in the  $s, \rho, \phi$  variables (the full form of the operator is long and will not be given here). The scale invariant solution take on the form  $s^{\zeta_3/2} f(\rho, \phi)$ , and the transformed operator applied to this form gives an equation for  $f(t, \phi)$

$$\hat{B}_3(\zeta_3) f(\rho, \phi) = [a(\rho, \phi) \partial_\rho^2 + b(\rho, \phi) \partial_\phi^2 + c(\rho, \phi) \partial_\rho \partial_\phi \\ + u(\rho, \phi, \zeta_3) \partial_\rho + v(\rho, \phi, \zeta_3) \partial_\phi + w(\rho, \phi, \zeta_3)] f(\rho, \phi) = 0. \quad (9)$$

The new operator  $\hat{B}_3$  depends on  $\zeta_3$  as a parameter and it acts on the unit circle described by the polar  $\rho, \phi$  coordinates. The circle represents the projective space of the physical cone described above.

The discrete permutation symmetry of the original Eq.(6) results in a symmetry of Eq.(9) with respect to the 6 element group generated by the transformation  $\phi \rightarrow \phi + 2\pi/3$  (cyclic permutation of the coordinates in the physical space) and  $\phi \rightarrow -\phi$  (exchange of coordinates). This symmetry extends to a full  $U(1)$  symmetry in the two marginal cases of  $\zeta_h = 0$  and  $\zeta_h = 2$  for which all the coefficients in (9) become  $\phi$ -independent. The coefficients in (9) all have a similar structure, and for example  $a(\rho, \phi)$  reads

$$a(\rho, \phi) = \sum_n [1 - \rho \cos(\phi + \frac{2}{3}\pi n)]^{(\zeta_h - 2)/2} \tilde{a}(\rho, \phi + \frac{2}{3}\pi n),$$

where  $\tilde{a}(\rho, \phi)$  is a low order polynomial in  $\rho$ ,  $\cos \phi$  and  $\sin \phi$  which vanishes at  $\rho = 1, \phi = 0$ . We see that the coefficients are analytic everywhere

FIG. 1. The scaling exponent  $\zeta_3$  as a functions of  $\zeta_h$  found as the loci of zeros of the determinant of the matrix  $B_3$ , for  $d = 2$ .

on the circle except at the three points  $\rho = 1$ ,  $\phi = 2\pi n/3$  where  $n = 0, 1, 2$ . These points correspond to the fusion of one pair of coordinates, and the coefficients exhibit a branch point singularity there. This singularity leads to a nontrivial asymptotic behavior of the solutions which had been described before in terms of the fusion rules [6,11]. Note that for  $\zeta_h = 2$  the singularity disappears trivially. For  $\zeta_h = 0$  there is also no singularity since  $\tilde{a}$  exactly compensates for the inverse power.

The boundary conditions follow naturally when one realizes that  $\hat{B}_3$  is elliptic for points strictly inside the physical circle. This is a consequence of the ellipticity of the original operator  $\hat{B}_3$ . On the other hand  $\hat{B}_3$  becomes singular on the boundary  $\rho = 1$ , where the coefficients  $a(\rho, \phi)$  and  $c(\rho, \phi)$  vanish. This singularity reflects the fact that this is the boundary of the physical region. It follows that  $\hat{B}_3$  restricted to the boundary becomes a relation between the function  $f(\rho = 1, \phi) \equiv g(\phi)$  and its normal derivative  $\partial_\rho f(\rho, \phi)|_{\rho=1} \equiv h(\phi)$ . The relation is  $bg'' + uh + vg' + wg = 0$ . Solutions of Eq.(9) which do not satisfy this boundary condition are singular, with infinite  $\rho$  derivatives at  $\rho = 1$ . Such solutions are not physical since they involve infinite correlations between the dissipation (second derivative of the field) and the field itself when the geometry becomes collinear, but without fusion.

Having a homogeneous equation with homogeneous boundary conditions we realize that non-trivial solutions are available only when  $\det(\hat{B}_3) = 0$ . This determinant depends parametrically on  $\zeta_3$ . Since the operator is defined on a compact domain we expect the determinant to vanish only for discrete values of  $\zeta_3$  for any given values of  $\zeta_h$  and the dimensionality  $d$ . We know that there always exists a trivial constant solution associated with  $\zeta_3 = 0$ . Our aim is to find the lowest lying positive real solutions  $\zeta_3$  for which the determinant vanishes.

FIG. 2. Same as Fig.1, but for  $d = 3$ .

FIG. 3. Same as Fig.1, but for  $d = 4$ .

We approach the problem numerically by discretizing the operator  $\hat{B}_3$  including the boundary conditions, and solving the analogous problem for the discretized operator. Using the symmetry of the problem we restricted the domain to one sixth of the circle, and defined a nine-point finite difference scheme for the evaluation of the second order derivatives. The discretized boundary conditions at  $\rho = 1$  were achieved with the same scheme. The symmetry implies that the new boundary conditions on the lines  $\phi = 0, \pi/3$  are simple Neuman boundary conditions

$\partial_\phi f(\rho, \phi) = 0$ . After discretization the problem transforms to a matrix eigenvalue problem  $B_3 \Psi = 0$ , where  $B_3$  is a large sparse matrix, whose rank depends on the mesh of the discretization, and  $\Psi$  is the discretized  $f$ . We used NAG's sparse Gaussian elimination routines to find the zeros of  $\det(B_3)$ , and determined the values of  $\zeta_3$  for these zeros as a function of  $\zeta_h$ . The results of this procedure for space dimensions  $d = 2, 3, 4$  are presented in Figs. 1,2, and 3.

The various branches shown in Figs. 1-3 can be organized on the basis of the perturbation theory of the type proposed in [3] near  $\zeta_h = 0$ . We performed that type of analysis and found that at  $\zeta_h = 0$  the allowed values of  $\zeta_3$  are organized in two sets,

$$\begin{aligned}\zeta_3^+(m, n) &= 2(3m + 2n), \\ \zeta_3^-(m, n) &= -2(d - 1 + 3m + 2n),\end{aligned}\quad (10)$$

where  $n$  and  $m$  are any non-negative integer. The lowest lying positive values are 4, 6, 8 etc, whereas for  $d = 2$  the highest negative value is  $-2$ . We see that the non-perturbative solution displays in all dimensions a branch (dashed line) which begins at  $\zeta_h = 0, \zeta_3 = 4$  and ends at  $\zeta_h = 2, \zeta_3 = 0$ . This branch is identical to the lowest lying positive branch predicted by the perturbation theory. We computed the slope of this branch near  $\zeta_h = 0$  in perturbation theory, and found that it is  $2(2-d)/(d-1)$ , in agreement with the numerics. Also the slopes of the other branches that begin at  $\zeta_h = 0$  were obtained perturbatively and found to agree with the numerics. The negative branch (shown only for  $d = 2$ ) never rises above its perturbative limit and is not relevant for the scaling behaviour at any value of  $\zeta_h$ . Note also that the point  $\zeta_h = 2, \zeta_3 = 0$  appears to be an accumulation point of many branches, and we are not confident that all the branches there were identified by our finite discretization scheme. This raises a worry about the availability of a smooth perturbative theory around  $\zeta_h = 2$ . At least we expect such a perturbation theory to be very singular. Preliminary analytical work indicates that all the branches join the point  $\zeta_h = 2, \zeta_3 = 0$  with an infinite slope.

The results of our nonperturbative approach lend support to the validity of the perturbative calculations of the zero modes of  $\hat{B}_4$ . The disagreement between the scaling exponents  $\zeta_4$  and the higher order exponents  $\zeta_n$  computed via the perturbative approach and the predictions of the other approach based on the fully fused theory cannot be ascribed to a formal failure of the perturbation theory. There are therefore a few possibilities that have to be sorted out by further research:

- (i) The crucial assumption that goes to the fully fused approach, which is the linearity of the conditional average of the Laplacian of the scalar, is wrong.
- (ii) The computation of the zero modes which is achieved by discarding the viscous terms in  $\hat{B}_n$  is irrelevant for the

physical solution. It is not impossible that the diffusive term act as a singular perturbation on some of the scale invariant modes. In fact, the operator  $\hat{B}_n$  with the viscous term is positive definite and it has no zero modes. That this is a possibility is underlined by recent calculations of a shell model of the Kraichnan model [13], in which it was shown that the addition of any minute diffusivity changes the nature of the zero modes qualitatively. (iii) Lastly, and maybe most interestingly, it is possible that the physical solution is not scale invariant [12]. In other words, it is possible that  $\mathcal{F}_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$  is not a homogeneous functions with a fixed homogeneity exponent  $\zeta_3$ , but rather (for example), that  $\zeta_3$  depends on the ratios of the separations (or, in other words, the geometry of the triangle defined by the coordinates). If this were also the case for even correlation functions  $\mathcal{F}_{2n}$ , this would open an exciting route for further research to understand how non-scale invariant correlation functions turn, upon fusion, to scale invariant structure functions.

In light of the numerical results of ref. [5] and the experimental results displayed in [7,8] we tend to doubt option (i). If we were to guess at this point we would opt for possibility (ii). More work however is needed to clarify this important issue beyond doubt.

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